Math 055 Exam 2 Study Guide

This study guide is designed to help you prepare specifically for Exam 2. Every item below corresponds directly to skills and ideas from second-order and higher-order differential equations and their applications.

1. Linear Differential Equations of Higher Order (Homogeneous)

What you should be able to do:

- Solve constant-coefficient linear differential equations of order 2 and higher.
- Form and solve the characteristic equation.
- Handle:
  - Distinct real roots
  - Repeated roots
  - Complex roots

You should understand:

- Why the form of the solution depends on the type of roots.
- How linear independence affects the structure of the general solution.

Practice focus: Solve a third-order linear homogeneous differential equation with mixed real and complex roots.

2. Nonhomogeneous Equations: Method of Undetermined Coefficients

What you should be able to do:

- Identify when undetermined coefficients is appropriate.
- Choose the correct form of a particular solution for:
  - Polynomials
  - Exponentials
  - Trigonometric functions
- Modify the guess when overlap (resonance) occurs.
- Solve for unknown coefficients.

You should understand:

- Why the form of the guess mirrors the forcing function.

- How resonance affects the structure of the trial solution.

Practice focus: Solve nonhomogeneous equations with different types of forcing functions and determine when modification of the guess is required.

3. Cauchy–Euler Equations

What you should be able to do:

- Recognize equations of the form $ax^2y'' + bxy' + cy = 0$.
- Use the substitution $y = x^r$ to find solutions.
- Solve the resulting algebraic equation.
- Write the general solution for real and complex roots.

You should understand:

- Why powers of x are natural solutions for this type of equation.
- The connection between this method and constant-coefficient equations.

Practice focus: Solve a Cauchy–Euler equation with complex roots.

4. Reduction of Order

What you should be able to do:

- Find a second linearly independent solution when one solution is given.
- Use either:
  - The substitution $y_2 = v(x)y_1$, or
  - The reduction of order formula (after putting the equation in standard form).

You should understand:

- Why a second independent solution is needed for the general solution.
- How the structure of the differential equation allows order reduction.

Practice focus: Given one solution to a second-order equation, find a second solution using reduction of order.

5. Variation of Parameters

What you should be able to do:

- Use a known complementary solution to construct a particular solution.
- Compute the Wronskian.
- Set up and evaluate the integrals for the parameter functions.
- Assemble the full particular solution.

You should understand:

- How this method differs from undetermined coefficients.
- Why it works for a broader class of forcing functions.

Practice focus: Use variation of parameters to solve a nonhomogeneous second-order differential equation.

6. Boundary-Value Problems

What you should be able to do:

- Solve second-order differential equations with boundary conditions.
- Apply conditions at two different points.
- Determine whether:
  - A unique solution exists
  - No solution exists
  - Infinitely many solutions exist

You should understand:

- How boundary conditions differ from initial conditions.
- Why certain conditions lead to no solution or non-uniqueness.

Practice focus: Solve a boundary-value problem and interpret the result.

7. Mechanical Vibrations (Mass–Spring Systems)

What you should be able to do:

- Convert physical information into a differential equation.
- Determine mass and spring constant from given data.
- Solve the resulting second-order equation.
- Express the solution in sinusoidal form.
- Find amplitude, period, frequency, and phase.

You should understand:

- The physical meaning of each parameter in the model.
- How initial conditions determine the motion.

Practice focus: Model and solve a mass–spring system with given initial conditions.

Practice Worksheet

1. Higher-Order Homogeneous Equation Solve the differential equation: $y''' + 2y'' - y' - 2y = 0$.

2. Undetermined Coefficients (Polynomial) Solve: $y'' - 3y = x^2 + 1$.

3. Undetermined Coefficients (Exponential) Solve: $y'' + y' - 2y = e^{2x}$.

4. Undetermined Coefficients (Trigonometric) Solve: $y'' + 4y = 3\cos(2x)$.

5. Cauchy–Euler Equation Solve: $x^2y'' + 3xy' + 2y = 0$.

6. Reduction of Order Given that $y_1 = e^x$ is a solution to $y'' - y = 0$, find a second solution y_2.

7. Variation of Parameters Find a particular solution of: $y'' - y = e^x$.

8. Boundary-Value Problem Solve the differential equation: $y'' + y = 0$, subject to: $y(0) = 2, \quad y\left(\frac{\pi}{2}\right) = 0$.

9. Mass–Spring System A mass weighing 4 pounds stretches a spring 1 foot. The mass is released from equilibrium with an upward velocity of 3 ft/s.
- Set up the differential equation.
- Solve for the position $x(t)$.
- Find the amplitude and period.

10. LRC-Series Circuit An LRC-series circuit has:
- $L = 2$ henries
- $R = 6$ ohms
- $C = \frac{1}{8}$ farads
- $E(t) = 0$

Initial conditions: $q(0) = 1, \quad q'(0) = 0$.

- Write the differential equation.
- Solve for the charge $q(t)$.
- Describe the behavior of the system.

Note: You are not responsible for remembering the DE for an LRC-series circuit.